

ON OPTIMAL HÖLDER REGULARITY OF SOLUTIONS TO THE EQUATION $\Delta u + b \cdot \nabla u = 0$ IN TWO DIMENSIONS

NAM Q. LE

ABSTRACT. We show that for an L^2 drift b in two dimensions, if the Hardy norm of $\operatorname{div} b$ is small, then the weak solutions to $\Delta u + b \cdot \nabla u = 0$ have the same optimal Hölder regularity as in the case of divergence-free drift, that is, $u \in C_{\operatorname{loc}}^\alpha$ for all $\alpha \in (0, 1)$.

1. STATEMENT OF THE MAIN RESULT

In this note, we revisit the local (optimal) Hölder continuity of $W^{1,2}$ scalar solutions to

$$(1.1) \quad \Delta u + b \cdot \nabla u = 0$$

in two dimensions where the drift $b = (b_1, b_2)$ is an L_{loc}^2 vectorfield in \mathbb{R}^2 . When $\operatorname{div} b = 0$, Filonov [3, Theorem 1.2] shows that $u \in W_{\operatorname{loc}}^{2,q}$ for all $q \in (1, 2)$ and hence the optimal Hölder regularity for u : $u \in C_{\operatorname{loc}}^\alpha$ for all $\alpha \in (0, 1)$. In this note, we show that the same conclusions hold if the $\mathcal{H}^1(\mathbb{R}^2)$ Hardy norm of $\operatorname{div} b$ is small. The Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ will be recalled in Section 2.

Theorem 1.1. *Let Ω be an open, bounded and connected domain in \mathbb{R}^2 . Let $b \in L_{\operatorname{loc}}^2(\mathbb{R}^2; \mathbb{R}^2)$ and let $u \in W^{1,2}(\Omega)$ satisfy (1.1) in Ω in the sense of distributions. There is a small, positive constant ε_0 such that if $\|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq \varepsilon_0$ then $u \in W_{\operatorname{loc}}^{2,q}(\Omega)$ for all $q \in (1, 2)$ and hence, $u \in C_{\operatorname{loc}}^\alpha(\Omega)$ for all $\alpha \in (0, 1)$.*

If the condition $\operatorname{div} b = 0$ is dropped, then, as pointed out in [3], the solutions of (1.1) are not continuous nor bounded in general. Note that, equation (1.1) with the divergence-free drift b appears in various models in fluid mechanics; see, for example [4, 9] and the references therein. These papers also establish several regularity results, including Hölder continuity, for solutions to (1.1) when the divergence-free drift b has low integrability. In 2D, in order to obtain the Hölder continuity of the solutions $u \in W^{1,2}$ to (1.1), that is, $u \in C_{\operatorname{loc}}^\alpha(\Omega)$ for some $\alpha \in (0, 1)$, it suffices to assume that $\operatorname{div} b \in \mathcal{H}^1(\mathbb{R}^2)$. This follows from revisiting the arguments of Bethuel [1].

Denote the rotation of the gradient vector in 2D by $\nabla^\perp v = (-\partial_2 v, \partial_1 v)$. Theorem 1.1 easily gives:

Corollary 1.2. *Let Ω be an open, bounded and connected domain in \mathbb{R}^2 . Assume that $b = h\nabla^\perp v$ where $h \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ and $v \in W^{1,2}(\Omega)$. Let $u \in W^{1,2}(\Omega)$ satisfy (1.1) in Ω . Then $u \in W_{\operatorname{loc}}^{2,q}(\Omega)$ for all $q \in (1, 2)$ and hence, $u \in C_{\operatorname{loc}}^\alpha(\Omega)$ for all $\alpha \in (0, 1)$.*

Remark 1.3. *The drift b in Corollary 1.2 satisfies $\operatorname{div} b \in \mathcal{H}^1$ and thus the result of Bethuel [1] (see also [5] for system) already implies the Hölder continuity of u , that is, $u \in C_{\operatorname{loc}}^\alpha(\Omega)$ for some $\alpha \in (0, 1)$. The novelty of Corollary 1.2 is that it gives further and optimal regularity results for u .*

Our proof of Theorem 1.1 uses the Uhlenbeck-Rivière decomposition (also known as the nonlinear Hodge decomposition) and the integration by compensation to convert (1.1) into a *conservation law*.

2010 *Mathematics Subject Classification.* 35B65, 35J15, 42B30.

Key words and phrases. Hardy space, Hölder continuity, nonlinear Hodge decomposition.

This circle of ideas was inspired by Rivière's proof of Heinz-Hildebrandt's conjecture [7]. We also give another proof of Theorem 1.1 using the uniqueness result for (1.1) following Filonov [3]. It is interesting to note that, *although (1.1) is linear, our arguments are non-linear*. A key ingredient in the proof of Theorem 1.1 is the local structure of the L^2 vectorfields b whose $\operatorname{div} b$ have small Hardy norm.

Theorem 1.4. *Assume that $\Omega = B_1(0) \subset \mathbb{R}^2$ and $b \in L^2_{loc}(\mathbb{R}^2; \mathbb{R}^2)$. There exists a positive constant ε_1 with the following property. If $\|b\|_{L^2(\Omega)} + \|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq \varepsilon_1$ then there exist $A \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $B \in W^{1,2}(\Omega)$ such that $A^{-1} \in L^\infty(\Omega)$, $A = 1$ on $\partial\Omega$, $\int_\Omega B = 0$, and*

$$b = A^{-1} \nabla A + A^{-1} \nabla^\perp B.$$

The rest of the note is organized as follows. We recall Hardy spaces and related Wente's estimates in Sect. 2. Assuming Theorem 1.4, we prove Theorem 1.1 in Sect. 3. We prove Theorem 1.4 in Sect. 4.

2. HARDY SPACES AND RELATED WENTE'S ESTIMATES

First, we recall the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$, following Hélein [6, Section 3.2]. For any function $f \in L^1(\mathbb{R}^2)$, we denote the Riesz transforms $R_j f$ ($j = 1, 2$) of f by $\widehat{R_j f}(\xi) = \frac{\xi_j}{|\xi|} \hat{f}(\xi)$ ($\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$) where \hat{f} is the Fourier transform of f : $\hat{f}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx$.

Definition 2.1. The Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ with norm $\|\cdot\|_{\mathcal{H}^1(\mathbb{R}^2)}$ is defined by

$$\mathcal{H}^1(\mathbb{R}^2) = \{f \in L^1(\mathbb{R}^2) : \|f\|_{\mathcal{H}^1(\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2)} + \|R_1 f\|_{L^1(\mathbb{R}^2)} + \|R_2 f\|_{L^1(\mathbb{R}^2)} < \infty\}.$$

A basic observation is the following theorem, due to Coifman-Lions-Meyer-Semmes [2]:

Theorem 2.2. ([6, Theorem 3.2.2]) *If $u, v \in W^{1,2}(\mathbb{R}^2)$ then $\nabla u \cdot \nabla^\perp v \in \mathcal{H}^1(\mathbb{R}^2)$. Moreover, there is a uniform constant C such that the following estimate holds:*

$$\|\nabla u \cdot \nabla^\perp v\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^2)} \|\nabla v\|_{L^2(\mathbb{R}^2)}.$$

We recall the following regularity result concerning solutions to the Laplace equation with Hardy right hand side. It follows from combining Theorems 3.3.4, 3.3.8 and together equation (3.38) in [6].

Theorem 2.3. *Let Ω be an open subset of \mathbb{R}^2 , with C^1 boundary. Let $f \in \mathcal{H}^1(\mathbb{R}^2)$, and $\phi \in L^1_{loc}(\Omega)$ be a solution of*

$$\Delta \phi = f \text{ in } \Omega, \text{ and } \phi = 0 \text{ on } \partial\Omega.$$

Then $\phi \in W^{1,2}_0(\Omega) \cap C(\overline{\Omega})$ and there is a constant depending only on Ω , $C(\Omega)$, such that

$$\|\phi\|_{L^\infty(\Omega)} + \|\nabla \phi\|_{L^2(\Omega)} \leq C(\Omega) \|f\|_{\mathcal{H}^1(\mathbb{R}^2)}.$$

Finally, we recall the following estimates for boundary value problems with Jacobian structure right hand side in the theory of integration by compensation, due to Wente's [10]; see also [2].

Lemma 2.4. *Let Ω be an open subset of \mathbb{R}^2 , with C^1 boundary. Suppose that $u, v \in W^{1,2}(\Omega)$. Let w be the unique solution in $W^{1,p}(\Omega)$ for $1 \leq p < 2$ to the equation $\Delta w = \nabla u \cdot \nabla^\perp v$ in Ω , either with the Dirichlet condition $w = 0$ on $\partial\Omega$, or with the Neumann boundary condition $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ and $\int_\Omega w = 0$. Then w belongs to $C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ and there is a constant C depending on Ω such that*

$$\|w\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^2(\Omega)} + \|D^2 w\|_{L^1(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}.$$

3. PROOFS OF THEOREM 1.1 AND COROLLARY 1.2

Assuming Theorem 1.4, we prove Theorem 1.1 and Corollary 1.2 in this section. Let ε_1 be the positive number given by Theorem 1.4. Let $\varepsilon_0 = \varepsilon_1/2$. Let $u \in W^{1,2}(\Omega)$ satisfy (1.1) in Ω .

3.1. Rescaling. Theorem 1.1 and Corollary 1.2 are of local nature so it suffices to prove the optimal Hölder continuity of u in a small ball $B_r(x_0) \subset \Omega$ round each $x_0 \in \Omega$. We rescale the equation (1.1) in $B_r(x_0)$ to $B_1(0)$ by setting

$$\tilde{u}(x) = u(x_0 + rx), \quad \tilde{b}(x) = rb(x_0 + rx).$$

Then $\tilde{u} \in W^{1,2}(B_1(0))$ solves

$$(3.2) \quad -\Delta \tilde{u} = \tilde{b} \cdot \nabla \tilde{u} \text{ in } B_1(0).$$

Furthermore,

$$(3.3) \quad \|\tilde{b}\|_{L^2(B_1(0))} = \|b\|_{L^2(B_r(x_0))} \text{ and } \|\operatorname{div} \tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} = \|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)}.$$

By the Dominated Convergence theorem, $\|b\|_{L^2(B_r(x_0))} \rightarrow 0$ as $r \rightarrow 0$. Thus, we can fix a radius $r > 0$ so that $\|b\|_{L^2(B_r(x_0))} < \varepsilon_0$.

From now on, we can assume $\Omega = B_1(0)$ with the following smallness condition on $\|\tilde{b}\|_{L^2(B_1(0))}$:

$$(3.4) \quad \|\tilde{b}\|_{L^2(B_1(0))} < \varepsilon_0.$$

3.2. Proofs of Theorem 1.1.

Proof of Theorem 1.1 via conservation laws. Suppose that $\|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq \varepsilon_0$. Then from (3.3) and (3.4), we have

$$\|\tilde{b}\|_{L^2(B_1(0))} + \|\operatorname{div} \tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq 2\varepsilon_0 = \varepsilon_1.$$

Applying Theorem 1.4, we find $A \in W^{1,2}(B_1(0)) \cap L^\infty(B_1(0))$ and $B \in W^{1,2}(B_1(0))$ such that $A^{-1} \in L^\infty(B_1(0))$, $A = 1$ on $\partial B_1(0)$, $\int_{B_1(0)} B = 0$, and

$$\tilde{b} = A^{-1} \nabla A + A^{-1} \nabla^\perp B.$$

This together with (3.2) gives

$$\operatorname{div}(A \nabla \tilde{u} - B \nabla^\perp \tilde{u}) = 0.$$

Thus, we have just converted (1.1) into a conservation law. By [8, Theorem 4.3] or the proof of [7, Theorem 1.1], $\tilde{u} \in W_{\operatorname{loc}}^{1,p}(B_1(0))$ for all $p \in (1, \infty)$. Now, the right hand side of (3.2) belongs to $L_{\operatorname{loc}}^q(B_1(0))$ for all $q \in (1, 2)$ and hence $\tilde{u} \in W_{\operatorname{loc}}^{2,q}(B_1(0))$ for all $q \in (1, 2)$. Therefore, by the Sobolev embedding theorem, $\tilde{u} \in C_{\operatorname{loc}}^\alpha(B_1(0))$ for all $\alpha \in (0, 1)$. Rescaling back, we obtain the desired regularity for u . \square

Proof of Theorem 1.1 via uniqueness. Instead of using [7, 8], we can give a direct and short proof of Theorem 1.1 using the uniqueness approach of Filonov [3]. In [3, Theorem 1.2], Filonov proved¹ that $W^{1,2}$ solutions of (3.2) belong to $W_{\operatorname{loc}}^{2,q}(B_1(0))$ for all $1 < q < 2$ provided that the equation

$$(3.5) \quad \Delta v + \tilde{b} \cdot \nabla v = 0 \text{ in } B_1(0), \text{ with } v = 0 \text{ on } \partial B_1(0).$$

has a unique solution $v = 0$ in $W_0^{1,2}(B_1(0))$. We prove that this is indeed the case in the context of Theorem 1.1. As above, we have $\tilde{b} = A^{-1}(\nabla A + \nabla^\perp B)$ and hence (3.5) becomes

$$(3.6) \quad \begin{cases} \operatorname{div}(A \nabla v) + \nabla^\perp B \cdot \nabla v &= 0 & \text{in } B_1(0), \\ v &= 0 & \text{on } \partial B_1(0). \end{cases}$$

¹Our sign is different from [3]. Filonov considered $-\Delta v + \tilde{b} \cdot \nabla v = 0$.

Since $A^{-1} \in L^\infty(B_1(0))$, we prove that $v = 0$ by showing $\int_{B_1(0)} A|\nabla v|^2 = 0$. Following the idea of the proof of [3, Lemma 2.6], we choose a sequence $\psi_n \in C_0^\infty(B_1(0))$ such that $\psi_n \rightarrow v$ in $W^{1,2}(B_1(0))$. From

$$\begin{aligned} \int_{B_1(0)} A|\nabla v|^2 &= \int_{B_1(0)} A\nabla v \cdot \nabla \psi_n + \int_{B_1(0)} A\nabla v \cdot (\nabla v - \nabla \psi_n) \\ &\leq \int_{B_1(0)} A\nabla v \cdot \nabla \psi_n + \|A\|_{L^\infty(B_1(0))} \|\nabla v\|_{L^2(B_1(0))} \|\nabla v - \nabla \psi_n\|_{L^2(B_1(0))}, \end{aligned}$$

and $\|\nabla v - \nabla \psi_n\|_{L^2(B_1(0))} \rightarrow 0$ when $n \rightarrow \infty$, it remains to show that

$$(3.7) \quad \int_{B_1(0)} A\nabla v \cdot \nabla \psi_n \rightarrow 0 \text{ when } n \rightarrow \infty.$$

To see this, multiplying both sides of (3.6) by ψ_n and using that

$$\int_{B_1(0)} \nabla^\perp B \cdot \nabla \psi_n \psi_n = \int_{B_1(0)} \nabla^\perp B \cdot \nabla (\psi_n^2/2) = 0$$

which follows from integrating by parts and $\operatorname{div} \nabla^\perp B = 0$, we obtain

$$\int_{B_1(0)} A\nabla v \cdot \nabla \psi_n = \int_{B_1(0)} \nabla^\perp B \nabla v \psi_n = \int_{B_1(0)} \nabla^\perp B \cdot \nabla (v - \psi_n) \psi_n.$$

Since $\operatorname{div}(\nabla^\perp B) = 0$, by [3, Lemma 2.4], and the fact that $\psi_n \rightarrow v$ in $W^{1,2}(B_1(0))$, we have

$$\int_{B_1(0)} \nabla^\perp B \cdot \nabla (v - \psi_n) \psi_n \leq C \|\nabla B\|_{L^2(B_1(0))} \|\nabla v - \nabla \psi_n\|_{L^2(B_1(0))} \|\nabla \psi_n\|_{L^2(B_1(0))} \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Therefore, we obtain (3.7) and the proof of Theorem 1.1 is complete. \square

Proof of Corollary 1.2. Suppose that $b = h\nabla^\perp v$ where $h \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ and $v \in W^{1,2}(\Omega)$. Corresponding to the rescalings of u and b in Section 3.1, we also rescale h and v as follows:

$$\tilde{h}(x) = h(x_0 + rx) - c_1, \tilde{v}(x) = v(x_0 + rx) - c_2$$

where c_1 and c_2 are constants so that $\int_{B_1(0)} \tilde{h} = \int_{B_1(0)} \tilde{v} = 0$. Note that $\|\nabla \tilde{v}\|_{L^2(B_1(0))} = \|\nabla v\|_{L^2(B_r(x_0))}$ and by Poincaré's inequality, we have

$$(3.8) \quad \|\tilde{v}\|_{W^{1,2}(B_1(0))} \leq C \|\nabla \tilde{v}\|_{L^2(B_1(0))} \leq C \|\nabla v\|_{L^2(B_r(x_0))}.$$

Similarly, we have

$$(3.9) \quad \|\tilde{h}\|_{W^{1,2}(B_1(0))} \leq C \|\nabla \tilde{h}\|_{L^2(B_r(x_0))}$$

We can extend \tilde{h} and \tilde{v} to be compactly supported functions in \mathbb{R}^2 such that $\tilde{h} \in L^\infty(\mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^2)$, $\tilde{v} \in W^{1,2}(\mathbb{R}^2)$, $\|\tilde{h}\|_{L^\infty(\mathbb{R}^2)} \leq C \|\tilde{h}\|_{L^\infty(B_1(0))}$, and

$$(3.10) \quad \|\tilde{h}\|_{W^{1,2}(\mathbb{R}^2)} \leq C \|\tilde{h}\|_{W^{1,2}(B_1(0))}, \quad \|\tilde{v}\|_{W^{1,2}(\mathbb{R}^2)} \leq C \|\tilde{v}\|_{W^{1,2}(B_1(0))}.$$

With these extensions, we have by Theorem 2.2, $\operatorname{div} \tilde{b} = \nabla \tilde{h} \cdot \nabla^\perp \tilde{v} \in \mathcal{H}^1(\mathbb{R}^2)$ and

$$\|\operatorname{div} \tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C \|\nabla \tilde{h}\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{v}\|_{L^2(\mathbb{R}^2)}.$$

It follows from (3.8), (3.9) and (3.10) that

$$\|\operatorname{div} \tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C \|\nabla \tilde{h}\|_{L^2(B_r(x_0))} \|\nabla \tilde{v}\|_{L^2(B_r(x_0))}.$$

Using the Dominated Convergence theorem, we can now further reduce the small radius r in Section 3.1 so that $\|\operatorname{div} \tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq \varepsilon_0$. Applying Theorem 1.1 to (3.2), we obtain the conclusion of the corollary. \square

4. PROOF OF THEOREM 1.4

4.1. Uhlenbeck-Rivière decomposition. Recall that $\Omega = B_1(0)$ and $\operatorname{div} b \in \mathcal{H}^1(\mathbb{R}^2)$. Let $\tau = (-y, x)$, and $\nu = (x, y)$ denote the unit tangential and normal vectorfields on $\partial\Omega$. We use the Hodge decomposition

$$(4.11) \quad b = \nabla^\perp \xi - \nabla p, \text{ where } p = 0 \text{ on } \partial\Omega.$$

To do this, let $p \in L^1_{\text{loc}}(\Omega)$ solve

$$-\Delta p = \operatorname{div} b \text{ in } \Omega, \quad p = 0 \text{ on } \partial\Omega.$$

Because $\operatorname{div} b \in \mathcal{H}^1(\mathbb{R}^2)$, by Theorem 2.3, we have $p \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Furthermore,

$$(4.12) \quad \|p\|_{L^\infty(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \leq C \|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)}.$$

With the above p , we have $\operatorname{div}(b + \nabla p) = 0$ so we can find $\xi \in W^{1,2}(\Omega)$ such that (4.11) holds. Now, suppose that we have a smallness condition on b , precisely, for some small $\varepsilon_1 > 0$ to be determined,

$$(4.13) \quad \|b\|_{L^2(\Omega)} + \|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq \varepsilon_1.$$

From (4.12) and (4.11), we have

$$(4.14) \quad \|p\|_{L^\infty(\Omega)}^2 + \int_\Omega |\nabla \xi|^2 + \int_\Omega |\nabla p|^2 \leq C \left(\int_\Omega |b|^2 + \|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)}^2 \right).$$

Inspired by Rivière [7, 8], we now rewrite (4.11) into the Uhlenbeck-Rivière decomposition (also known as the nonlinear Hodge decomposition). Let $P = e^p$. Then (4.11) becomes a nonlinear decomposition

$$(4.15) \quad b = \nabla^\perp \xi - P^{-1} \nabla P.$$

We will use (4.15) to convert (1.1) into a *conservation law*. Note that, $\nabla P = e^p \nabla p$, $\nabla P^{-1} = -e^{-p} \nabla p$. Let $\varepsilon \in (0, \frac{1}{100})$ be a small constant to be chosen in Lemma 4.1 below. With this ε , we choose ε_1 small so that from (4.13) and (4.14), we have

$$(4.16) \quad \int_\Omega |\nabla \xi|^2 + |\nabla P|^2 + |\nabla P^{-1}|^2 < \varepsilon$$

and

$$(4.17) \quad \|P\|_{L^\infty(\Omega)} \leq 1 + \varepsilon.$$

It follows that

$$(4.18) \quad 1/10 \leq \|P\|_{L^\infty(\Omega)}, \|P^{-1}\|_{L^\infty(\Omega)} \leq 10.$$

Recall that $P = 1$ on $\partial\Omega$. To prove Theorem 1.4, it remains to prove the following lemma.

Lemma 4.1. *Assume that (4.16) and (4.17) hold. If ε is sufficiently small then there exist $A \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $B \in W^{1,2}(\Omega)$ such that $A^{-1} \in L^\infty(\Omega)$, $A = 1$ on $\partial\Omega$, $\int_\Omega B = 0$, and*

$$(4.19) \quad Ab = \nabla A + \nabla^\perp B,$$

with

$$\|AP - 1\|_{L^\infty(\Omega)}^2 + \|\nabla(AP)\|_{L^2(\Omega)}^2 + \|\nabla B\|_{L^2(\Omega)}^2 \leq C\varepsilon.$$

4.2. Proof of Lemma 4.1.

Proof of Lemma 4.1. Notice that, by an approximation argument using the standard mollifications, it suffices to prove the lemma for smooth vectorfields b . In this case, we have on $\partial\Omega$

$$(4.20) \quad b \cdot \tau = (\nabla^\perp \xi - \nabla p) \cdot \tau = \frac{\partial \xi}{\partial \nu}.$$

The function ξ in (4.11) can be chosen to be the smooth solution to

$$-\Delta \xi = \operatorname{curl} b = \partial_2 b_1 - \partial_1 b_2 \quad \text{in } \Omega, \quad \frac{\partial \xi}{\partial \nu} = b \cdot \tau \quad \text{on } \partial\Omega \quad \text{and} \quad \int_{\Omega} \xi = 0.$$

In what follows, we will use equation (4.20). We use a fixed point argument as in Rivière [7, 8]. Let P be as in Sect. 4.1. To each $A \in W^{1,2}(\Omega)$, we associate $\tilde{A} = AP$. Suppose that A and B are solutions of (4.19). Then, recalling (4.15), and noting that $\nabla P P^{-1} = -P \nabla P^{-1}$, we have

$$\nabla \tilde{A} - \tilde{A} \nabla^\perp \xi + \nabla^\perp B \cdot P = 0 \quad \text{and} \quad \nabla^\perp B = A b - \nabla A = A \nabla^\perp \xi + A P \nabla P^{-1} - \nabla A.$$

Taking the divergence of the first equation and taking the curl ($= -\nabla^\perp$) of the second equation yield

$$\Delta \tilde{A} = \nabla \tilde{A} \cdot \nabla^\perp \xi - \nabla^\perp B \cdot \nabla P \quad \text{and} \quad \Delta B = \nabla^\perp (A \nabla^\perp \xi + A P \nabla P^{-1}) = \operatorname{div} (A \nabla \xi) + \nabla^\perp \tilde{A} \cdot \nabla P^{-1}.$$

We now proceed as follows.

Step 1. We prove, provided ε is sufficiently small, the existence of a solution (\tilde{A}, B) of the system

$$(4.21) \quad \begin{cases} \Delta \tilde{A} = \nabla \tilde{A} \cdot \nabla^\perp \xi - \nabla^\perp B \cdot \nabla P & \text{in } \Omega, \\ \Delta B = \operatorname{div} (\tilde{A} \nabla \xi P^{-1}) + \nabla^\perp \tilde{A} \cdot \nabla P^{-1} & \text{in } \Omega, \\ \tilde{A} = 1 \text{ and } \frac{\partial B}{\partial \nu} = b \cdot \tau & \text{on } \partial\Omega, \\ \int_{\Omega} B = 0, \end{cases}$$

with

$$\|\tilde{A} - 1\|_{L^\infty(\Omega)}^2 + \|\nabla \tilde{A}\|_{L^2(\Omega)}^2 + \|\nabla B\|_{L^2(\Omega)}^2 \leq C\varepsilon.$$

Step 2. We show that (4.21) implies

$$(4.22) \quad \nabla \tilde{A} - \tilde{A} \nabla^\perp \xi + \nabla^\perp B \cdot P = 0.$$

Let us indicate how Steps 1 and 2 complete the proof of Lemma 4.1. Assuming (4.22), we find from $\tilde{A} = AP$ that $P \nabla A + A \nabla P - A P \nabla^\perp \xi + \nabla^\perp B \cdot P = 0$. Since P is invertible, we obtain

$$\nabla A + A \nabla P P^{-1} - A \nabla^\perp \xi + \nabla^\perp B = 0.$$

Therefore, recalling (4.15), we obtain (4.19). The last estimate in Lemma 4.1 follows from the last estimate in Step 1 and the fact that $\tilde{A} = AP$. The proof of Lemma 4.1 is complete.

Proof of Step 1. To prove the existence of a solution (\tilde{A}, B) of (4.21), we will use a fixed point argument as in Rivière [7, 8]. Let us denote for $g \in H^{1/2}(\partial\Omega)$ the space $W_g^{1,2}(\Omega) = \{u \in W^{1,2}(\Omega), u = g \text{ on } \partial\Omega\}$.

Consider the map $f(\hat{A}, \hat{B}) = (\tilde{A}, B)$ from

$$X = \left(W_1^{1,2}(\Omega) \cap L^\infty(\Omega) \right) \times W^{1,2}(\Omega)$$

into itself, where for given $(\hat{A}, \hat{B}) \in X$, the pair (\tilde{A}, B) solves the system

$$(4.23) \quad \begin{cases} \Delta(\tilde{A} - 1) = \nabla(\hat{A} - 1) \cdot \nabla^\perp \xi - \nabla^\perp \hat{B} \cdot \nabla P & \text{in } \Omega, \\ \Delta(B - B_0) = \operatorname{div} ((\hat{A} - 1) \nabla \xi P^{-1}) + \nabla^\perp (\hat{A} - 1) \cdot \nabla P^{-1} & \text{in } \Omega, \\ \tilde{A} = 1 \text{ and } \frac{\partial B}{\partial \nu} = b \cdot \tau & \text{on } \partial\Omega, \\ \int_{\Omega} B = 0. \end{cases}$$

Here, the function $B_0 \in W^{1,2}(\Omega)$ is the solution to

$$(4.24) \quad \begin{cases} \Delta B_0 = \operatorname{div}(\nabla \xi P^{-1}) & \text{in } \Omega, \\ \frac{\partial B_0}{\partial \nu} = b \cdot \tau & \text{on } \partial\Omega, \\ \int_{\Omega} B_0 = 0. \end{cases}$$

Clearly, a fixed point of (4.23) is a solution of (4.21).

By (4.20), the equation (4.24) has a unique solution. Multiplying both sides of the first equation of (4.24) by B_0 , integrating by parts, we find from $P = 1$ on $\partial\Omega$ and (4.20) that $\int_{\Omega} |\nabla B_0|^2 = \int_{\Omega} \nabla \xi P^{-1} \cdot \nabla B_0$ from which we can estimate the gradient of B_0 by

$$(4.25) \quad \int_{\Omega} |\nabla B_0|^2 \leq \int_{\Omega} |P^{-1} \nabla \xi|^2 \leq \|P^{-1}\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla \xi|^2.$$

Applying Lemma 2.4 to $\tilde{A} - 1$ and recalling the first and third equations in (4.23), we find

$$(4.26) \quad \|\tilde{A} - 1\|_{L^\infty(\Omega)}^2 + \|\nabla \tilde{A}\|_{L^2(\Omega)}^2 \leq C \int_{\Omega} |\nabla \hat{A}|^2 \int_{\Omega} |\nabla \xi|^2 + C \int_{\Omega} |\nabla \hat{B}|^2 \int_{\Omega} |\nabla P|^2.$$

Note that, by the second and last equations in (4.23), $B - B_0 \in W^{1,2}(\Omega)$ satisfies

$$(4.27) \quad \begin{cases} \Delta(B - B_0) = \operatorname{div}((\hat{A} - 1)\nabla \xi P^{-1}) + \nabla^\perp \hat{A} \cdot \nabla P^{-1} & \text{in } \Omega, \\ \frac{\partial(B - B_0)}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} (B - B_0) = 0. \end{cases}$$

At this point, we can use Theorem 2.4 and argue as in (4.25) to obtain the estimate

$$(4.28) \quad \|\nabla(B - B_0)\|_{L^2(\Omega)}^2 \leq C \|P^{-1}\|_{L^\infty(\Omega)}^2 \|\hat{A} - 1\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla \xi|^2 + C \int_{\Omega} |\nabla \hat{A}|^2 \int_{\Omega} |\nabla P^{-1}|^2.$$

This combined with (4.25) gives

$$(4.29) \quad \begin{aligned} \|\nabla B\|_{L^2(\Omega)}^2 &\leq C \|P^{-1}\|_{L^\infty(\Omega)}^2 \|\hat{A} - 1\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla \xi|^2 + C \int_{\Omega} |\nabla \hat{A}|^2 \int_{\Omega} |\nabla P^{-1}|^2 \\ &\quad + C \|P^{-1}\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla \xi|^2. \end{aligned}$$

The above arguments show that for $(\hat{A}, \hat{B}), (\hat{A}_1, \hat{B}_1) \in X$, the pairs $(\tilde{A}, B) = f(\hat{A}, \hat{B}), (\tilde{A}_1, B_1) = f(\hat{A}_1, \hat{B}_1)$ satisfy the estimates

$$\|\tilde{A} - \tilde{A}_1\|_{L^\infty(\Omega)}^2 + \|\nabla(\tilde{A} - \tilde{A}_1)\|_{L^2(\Omega)}^2 \leq C \int_{\Omega} |\nabla(\hat{A} - \hat{A}_1)|^2 \int_{\Omega} |\nabla \xi|^2 + C \int_{\Omega} |\nabla(\hat{B} - \hat{B}_1)|^2 \int_{\Omega} |\nabla P|^2.$$

and

$$\|\nabla(B - B_1)\|_{L^2(\Omega)}^2 \leq C \|P^{-1}\|_{L^\infty(\Omega)}^2 \|(\hat{A} - \hat{A}_1)\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla \xi|^2 + C \int_{\Omega} |\nabla(\hat{A} - \hat{A}_1)|^2 \int_{\Omega} |\nabla P^{-1}|^2.$$

Since $\int_{\Omega} (B - B_1) = 0$, we have by the Poincaré inequality $\int_{\Omega} |B - B_1|^2 \leq C \int_{\Omega} |\nabla(B - B_1)|^2$. Hence, if $\int_{\Omega} |\nabla \xi|^2 + |\nabla P|^2 + |\nabla P^{-1}|^2 \leq \varepsilon$ is sufficiently small, then a standard fixed point argument in the space $X = (L^\infty(\Omega) \cap W_1^{1,2}(\Omega)) \times W^{1,2}(\Omega)$ yields the existence of a solution (\tilde{A}, B) of the system (4.21).

Furthermore, from (4.26), (4.29) together with (4.16) and (4.18), the solution (\tilde{A}, B) of the system (4.21) satisfies

$$\begin{aligned} \|\tilde{A} - 1\|_{L^\infty(\Omega)}^2 + \|\nabla \tilde{A}\|_{L^2(\Omega)}^2 &\leq C\varepsilon \|\nabla \tilde{A}\|_{L^2(\Omega)}^2 + C\varepsilon \|\nabla B\|_{L^2(\Omega)}^2 \\ &\leq C\varepsilon \|\nabla \tilde{A}\|_{L^2(\Omega)}^2 + C\varepsilon^2 (1 + \|\tilde{A} - 1\|_{L^\infty(\Omega)}^2 + \|\nabla \tilde{A}\|_{L^2(\Omega)}^2). \end{aligned}$$

Thus, if $C\varepsilon \leq 1/4$, we have

$$\|\tilde{A} - 1\|_{L^\infty(\Omega)}^2 + \|\nabla \tilde{A}\|_{L^2(\Omega)}^2 \leq C\varepsilon.$$

By (4.29), we then have $\|\nabla B\|_{L^2(\Omega)}^2 \leq C\varepsilon$. The proof of Step 1 is complete.

Proof of Step 2. To show (4.22), we introduce the Hodge decomposition

$$(4.30) \quad \nabla \tilde{A} - \tilde{A} \nabla^\perp \xi + \nabla^\perp B \cdot P = \nabla E + \nabla^\perp D$$

where $E = 0$ on $\partial\Omega$. Taking divergence of both sides of (4.30), and recalling the first equation of (4.21), we get $\Delta E = 0$ in Ω . Hence, $E \equiv 0$ and (4.30) reduces to

$$(4.31) \quad \nabla \tilde{A} - \tilde{A} \nabla^\perp \xi + \nabla^\perp B \cdot P = \nabla^\perp D.$$

It remains to show that D is a constant. By (4.31), and recalling (4.21), we have

$$\nabla D = -\nabla^\perp \tilde{A} - \tilde{A} \nabla \xi + P \nabla B \text{ and } \Delta D = P(\tilde{A} \nabla \xi \cdot \nabla P^{-1} + \nabla^\perp \tilde{A} \cdot \nabla P^{-1} + \nabla B \cdot \nabla P P^{-1}).$$

It follows from a simple calculation that

$$(4.32) \quad \operatorname{div}(\nabla D P^{-1}) = (-\nabla^\perp \tilde{A} - \tilde{A} \nabla \xi + P \nabla B) \cdot \nabla P^{-1} + \Delta D P^{-1} = \nabla B \cdot \nabla(P P^{-1}) = 0.$$

With (4.32), we complete the proof of Step 2 as follows. Taking dot product with τ on both sides of (4.31), and recalling that $P = \tilde{A} = 1$ on $\partial\Omega$, we find that on $\partial\Omega = \partial B_1(0)$:

$$D_\nu = \nabla^\perp D \cdot \tau = \tilde{A}_\tau - \tilde{A} \xi_\nu + B_\nu P = B_\nu - \xi_\nu = b \cdot \tau - b \cdot \tau = 0.$$

Multiplying both sides of (4.32) by D and integrating by parts, we find

$$0 = \int_\Omega \operatorname{div}(\nabla D \cdot P^{-1}) D = - \int_\Omega P^{-1} |\nabla D|^2 + \int_{\partial\Omega} P^{-1} D \nabla D \cdot \nu = - \int_\Omega P^{-1} |\nabla D|^2.$$

Recalling (4.18), we obtain $\nabla D = 0$ in Ω and hence D is a constant in Ω . \square

REFERENCES

- [1] Bethuel, F. Un résultat de régularité pour les solutions de l'équation de surfaces à courbure moyenne prescrite. *C. R. Acad. Sci. Paris Sér. I Math.* **314** (1992), no. 13, 1003–1007.
- [2] Coifman, R.; Lions, P.-L.; Meyer, Y.; Semmes, S. Compensated compactness and Hardy spaces. *J. Math. Pures Appl.* (9) **72** (1993), no. 3, 247–286.
- [3] Filonov, N. On the regularity of solutions to the equation $\Delta u + b \cdot \nabla u = 0$. *Journal of Mathematical Sciences* **195** (2013), no. 1, 98–108.
- [4] Friedlander, S.; Vicol, V. Global well-posedness for an advection-diffusion equation arising in magneto-geostrophic dynamics. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** (2011), no. 2, 283–301.
- [5] Hajlasz, P.; Strzelecki, P.; Zhong, X. A new approach to interior regularity of elliptic systems with quadratic Jacobian structure in dimension two. *Manuscripta Math.* **127** (2008), no. 1, 121–135.
- [6] Hélein, F. *Harmonic Maps, Conservation Laws and Moving Frames*, Cambridge Tracts in Math., vol. 150, Cambridge University Press, Cambridge, 2nd edition, 2002.
- [7] Rivière, T. Conservation laws for conformally invariant variational problems. *Invent. Math.* **168** (2007), no. 1, 1–22.
- [8] Rivière, T. The role of conservation laws in the analysis of conformally invariant problems. *Topics in modern regularity theory*, 117–167, CRM Series, 13, Ed. Norm., Pisa, 2012.
- [9] Seregin, G.; Silvestre, L.; Šverák, V.; Zlatoš, A. On divergence-free drifts. *J. Differential Equations* **252** (2012), no. 1, 505–540.
- [10] Wente, H. C. An existence theorem for surfaces of constant mean curvature. *J. Math. Anal. Appl.* **26** (1969) 318–344.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405, USA

E-mail address: nqle@indiana.edu